

GRAPHS OF MULTIFUNCTIONS

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1. INTRODUCTION

In [4] N. Shcherbina gives a positive answer to the following question of Nishino (see [2]):

Let \mathbb{D} be the unit disc in \mathbb{C} and let $f : \mathbb{D} \rightarrow \mathbb{C}$ be a continuous function such that its graph

$$(1) \quad \Gamma(f) = \{(z, f(z)) : z \in \mathbb{D}\}$$

is a pluripolar subset of \mathbb{C}^2 . Does it follow that f is holomorphic?

Recently, in [5] N. Shcherbina proves a more general result for multi-values functions defined by Weierstrass pseudopolynomial.

Theorem 1. *Let D be a domain in \mathbb{C}^n and let \mathcal{E} be a subset of $D \times \mathbb{C}$ of the form*

$$(2) \quad \mathcal{E} = \{(z, w) \in D \times \mathbb{C} : w^m + a_1(z)w^{m-1} + \cdots + a_m(z) = 0\},$$

where a_1, \dots, a_m are continuous complex-valued functions on D . Then \mathcal{E} is a pluripolar subset of \mathbb{C}^{n+1} if and only if the functions a_1, \dots, a_m are holomorphic on D .

The proof of Theorem 1, presented in [5], is based on Oka's result (see e.g. [3], Theorem 4.9) and is a generalization of the methods of [4].

The main purpose of the paper is to show that Theorem 1 is a simple corollary of the results of [4]. Actually, our proof is similiar, in spirit, to the proof of the Oka's result.

2. PROOFS

Recall the following result of Radó (see e.g. [1]).

Theorem 2. *Let D be a domain in \mathbb{C}^n and let f be a continuous function on D . Assume that f is holomorphic on $D \setminus f^{-1}\{0\}$. Then f is holomorphic on D .*

Proof of Theorem 1. It suffices to show that from pluripolarity of \mathcal{E} it follows holomorphicity of a_1, \dots, a_m . We prove by induction on m .

For $m = 1$ it's the main result in [4].

So, assume that Theorem 1 holds for any domain D and for $1, 2, \dots, m-1$. Put

$$P(z, w) = w^m + a_1(z)w^{m-1} + \cdots + a_m(z).$$

Fix $z_0 \in D$. Let w_1, \dots, w_s be all distinct solutions of the equation $P(z_0, \cdot) = 0$. Assume that $s > 1$ (i.e. the multiplicity at any point w_j is less than m).

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Let us show that a_1, \dots, a_m are holomorphic in a neighborhood of z_0 . Indeed, fix $j \in \{1, \dots, s\}$. Let k be the multiplicity of $P(z_0, \cdot) = 0$ at w_j (we know that $k < m$). Then there exists an $r > 0$ such that for any $z \in \mathbb{B}(z_0, r) = \{\zeta \in \mathbb{C}^n : \|\zeta - z_0\| < r\}$ in $\mathbb{D}(w_j, r) = \{\xi \in \mathbb{C} : |\xi - w_j| < r\}$ there is exactly k solutions of the equation $P(z, \cdot) = 0$ (counted with multiplicity). Moreover, there is no solution on $\partial\mathbb{D}(w_j, r)$. Assume that $\zeta_1(z), \dots, \zeta_k(z)$ are these solutions. Put

$$\tilde{P}(z, w) = (w - \zeta_1(z)) \dots (w - \zeta_k(z)).$$

Note \tilde{P} is well-defined on $\mathbb{B}(z_0, r)$ and that its coefficients are continuous in $\mathbb{B}(z_0, r)$. Since

$$\{(z, w) \in \mathbb{B}(z_0, r) \times \mathbb{C} : \tilde{P}(z, w) = 0\} \subset \mathcal{E},$$

by induction step we get that its coefficients are holomorphic in $\mathbb{B}(z_0, r)$.

We may repeat the same argument for all the solutions w_1, \dots, w_s of the equation $P(z_0, \cdot) = 0$ and get that a_1, \dots, a_m (as polynomials of the previously obtained coefficients) are holomorphic in some neighborhood of z_0 .

Let B denote the set of all points $z \in D$ such that $P(z, \cdot) = 0$ has a solution with the multiplicity m . Then B is a closed subset of D . Moreover, a_1, \dots, a_m are holomorphic on $D \setminus B$. If B is a pluripolar set, then a_1, \dots, a_m are holomorphic on D (use the continuity). So, assume that B is non-pluripolar.

Let

$$\varphi_k(z, w) = \frac{\partial^k P(z, w)}{\partial w^k}(z, w).$$

Note that $\varphi_{m-2}(z, w) = \frac{m!}{2}w^2 + (m-1)!a_1(z)w + (m-2)!a_2(z)$. Let

$$\Delta(z) = ((m-1)!a_1(z))^2 - 4\frac{m!}{2}(m-2)!a_2(z).$$

Note that $B \subset \{z \in D : \Delta(z) = 0\}$. Moreover, Δ is a continuous function on D and is holomorphic on $D \setminus \Delta^{-1}\{0\}$. Hence, by Radó's theorem, Δ is a holomorphic function on D . Since B is non-pluripolar, we have $\Delta \equiv 0$. So,

$$a_2(z) = \frac{m(m-1)}{2} \left(\frac{a_1(z)}{2} \right)^2.$$

Let us show by (inverse) induction on ℓ , that

$$\varphi_\ell(z, w) = \frac{m!}{(m-\ell)!} \left(w - \frac{a_1(z)}{m} \right)^{m-\ell}$$

for any $\ell = 1, \dots, m$. We know that it's true for $\ell = m, m-1, m-2$. Assume that it holds for $m, m-1, \dots, \ell+1$. We want to show that it holds for ℓ . By induction step

$$\begin{aligned} \varphi_\ell(z, w) &= \frac{m!}{(m-\ell)!} \left(w - \frac{a_1(z)}{m} \right)^{m-\ell} + (m-\ell)!a_\ell(z) \\ &\quad + (-1)^{m-\ell+1} \frac{m!}{(m-\ell)!} \left(\frac{a_1(z)}{m} \right)^{m-\ell}. \end{aligned}$$

Put $\psi_\ell = (m-\ell)!a_\ell(z) + (-1)^{m-\ell+1} \frac{m!}{(m-\ell)!} \left(\frac{a_1(z)}{m} \right)^{m-\ell}$. Note that ψ_ℓ is a continuous function and $B \subset \{z \in D : \psi_\ell(z) = 0\}$. Moreover, it is

holomorphic on $D \setminus \psi_\ell^{-1}\{0\}$. Hence, ψ_ℓ is holomorphic on D (again use Rado's theorem) and, therefore, $\psi_\ell \equiv 0$.

So, $B = D$ and $P(z, w) = (w - \frac{a_1(z)}{n})^m$. By induction step, a_1 (hence, a_2, \dots, a_m) is a holomorphic function. \square

Remark 3. Note that if in Theorem 1 there exists a point $z_0 \in D$ such that $P(z_0, \cdot)$ has m distinct solutions, then one can give even simpler proof. Indeed, put $\Delta(z) = \prod_{j \neq k} (\xi_j(z) - \xi_k(z))$, where $\xi_1(z), \dots, \xi_m(z)$ are the solutions (counted with multiplicity) of $P(z, \cdot) = 0$. Then Δ is a symmetric polynomial of $\xi_1(z), \dots, \xi_m(z)$, so it is a polynomial of $a_1(z), \dots, a_m(z)$. Therefore, Δ is a continuous function on D . Note that Δ and a_1, \dots, a_m are holomorphic functions on $D \setminus \Delta^{-1}\{0\}$ (use the results of [4]). We know that $\Delta(z_0) \neq 0$. So, $\Delta^{-1}(\{0\})$ is a proper analytic set. Since a_1, \dots, a_m are continuous on D and holomorphic on $D \setminus \Delta^{-1}\{0\}$, they are holomorphic on D .

Remark 4. N. Shcherbina informed me that he also found, independently, similar proof.

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